

# On the quasi-streamfunction formalism for waves and vorticity

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## Abstract

The quasi-streamfunction ( $\Psi$ ) formalism proposed by Kim et. al. (J.W. Kim, K.J. Bai, R.C. Ertekin, W.C. Webster, J. Eng. Math. 40, 17 (2001)) provides a natural framework for systematically studying zero-vorticity waves over arbitrary bathymetry. The modified  $\Psi$ -formalism developed here discards the original constraints of zero-vorticity by allowing for vertical vorticity which is the case of most interest for coastal dynamics. The problem is reformulated in terms of two dynamical equations on the boundary supplemented by one equation that represents a kinematic constraint in the interior of the domain. In this framework, the kinematic constraint can be solved to express  $\Psi$  in terms of the canonically-conjugated variables  $\eta$  and  $\phi$ . The formalism is demonstrated for horizontally homogeneous flows over mild topography, where asymptotic formulations for the Hamiltonian and Lagrangian functions are derived based on the Helmholtz-Hodge decomposition. For potential flows, the asymptotic form of the Hamiltonian is identical to previous results. The Lagrangian function is also expressed as an expansion in terms of measurable variables  $\eta$  and  $\partial_t \eta$ , and compared with Zakharov's formalism where agreement is found for one-dimensional wave scattering.

**Keywords:** stream function, variational principle, Hamiltonian function, Lagrangian function, Zakharov equation.

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## 1. Introduction

A complete understanding of wave evolution requires a consistent and unified formulation of the interaction of waves moving over varying topography with non-negligible shearing. Although

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numerical methods based on the Navier-Stokes equations are available, a simple lagrangian formalism that incorporates shearing seems to be lacking. In practice, one is often restricted to the assumption of gradient flow in order to approach coastal problems analytically. Another alternative, Clebsch variables [e.g., 1, 2], is a complete and consistent approach yet is physically unintuitive and not well suited to the study of wave phenomenon. Shallow water vorticity flows have also been studied using the shallow-water equation [3, 4]; described as a vortex sheet with constant vertical velocities [e.g., 5]; as thin shear layers flowing along discrete trenches [e.g., 6, 7]; and using a Hamiltonian formulation proposed by [8] for constant vorticity. However, these studies utilize a diverse variety of techniques and were targeted towards specific applications while a unifying formalism for arbitrary forcing and vorticity is still needed.

The quasi-streamfunction approach (hereafter denoted the ‘ $\Psi$ -formalism’) as studied here was proposed by Kim et al. [9, 10], and applied to surface waves by Kim and Bai [11] and Toledo and Agnon [12]. (The stream function  $\Psi$  was used much earlier by [13], in his study of nonlinear interactions of vortex filaments). Because the  $\Psi$ -formalism satisfies exactly the bottom boundary condition, it appears to provide simple way to systematically describe uneven topography. Although one is generally limited to the mild-slope approximation in solving the differential equations generated by the  $\Psi$  formalism, the method nevertheless gives one a convenient means for expanding solutions in terms of a mild-slope parameter. In addition, the formalism has the advantage of naturally allowing vertical vorticity; a feature which seems to have been overlooked in the original formulation [9, 10].

The purpose of this study is to remove the zero-vorticity restriction by incorporating a vertical vorticity. As the largest velocity components are horizontal in many oceanographic environments, vertical vorticity typically dominates over horizontal vorticity. Our formalism may be used, for instance, to study the propagation of waves through sheared currents or to study vorticity production in wave scattering. We hope to transform the  $\Psi$ -formalism into a more useful tool for the study of wave propagating over inhomogeneous topography and currents.

Some of the results here have been derived in Tian et. al. [14]. This paper presents an extension to their work. The governing equations are re-derived in their original Lagrangian formulation Section 2 and then solved using an expansion in powers of the free surface displacement  $\eta$  in Section 3. The derivation also provides an opportunity to correct a slight abuse of the variational princi-

ple in the original formulation. The Hamiltonian form of the  $\Psi$ -formalism is derived in Section 4. We propose a Lagrangian only containing measurable quantities  $\eta$  and  $\partial_t \eta$ , consistent with Zakharov's one-dimensional Hamiltonian formalism. The relationship between the  $\Psi$ -formalism and the potential flow is investigated in the Appendix, where we propose an interpretation of the function  $\Psi$ . Section 5 summarizes the results.

## 2. Governing Equations

Throughout this study,  $t$  denotes the time and the horizontal vectors are boldface, e.g.,  $\mathbf{x} = (x^1, x^2) = x^1 \hat{\mathbf{x}}_1 + x^2 \hat{\mathbf{x}}_2 = x^j \hat{\mathbf{x}}_j$ , with  $j = 1, 2$ . We will prefer Einstein's repeated-index summation convention (last equality). The origin of the coordinate system is set on the undisturbed free surface with the vertical axis ( $\hat{\mathbf{z}}$ ) pointing upward. The hat denotes a unit vector in the direction of the axis. The free surface is defined by  $z = \eta(\mathbf{x}, t)$  and the bottom by  $z = -h(\mathbf{x})$ . The symbol  $\nabla$  denotes the horizontal gradient.

Define the quasi-streamfunction as a vector:

$$\Psi(\mathbf{x}, z, t) = \int_{-h}^z \mathbf{u}(\mathbf{x}, s, t) ds. \quad (1)$$

The velocity field is defined as,

$$\mathbf{u} = \Psi_z, \quad (2)$$

where  $\mathbf{u}(\mathbf{x}, z, t) = u^j(\mathbf{x}, z, t) \hat{\mathbf{x}}_j$  is the horizontal velocity vector of components  $u^j$ , with  $j = 1, 2$ .

From the continuity of the incompressible fluid, we obtain the vertical velocity  $w$  as:

$$w = -\nabla \cdot \Psi. \quad (3)$$

Defining the total spatial gradient along a given surface  $z = \zeta(\mathbf{x}, t)$  as

$$D_\zeta = \nabla \cdot + (\nabla \zeta) \cdot \partial_z, \quad (4)$$

the total divergence of  $\Psi$  on the bottom  $z = -h$  is

$$D_{-h} \cdot \Psi = -w - (\nabla h) \cdot \mathbf{u} = 0, \quad (5)$$

which is the standard kinematic bottom boundary condition. This equality always holds because  $\Psi|_{z=-h} = 0$  by (1). Therefore the quasi-streamfunction  $\Psi$  unconditionally satisfies the kinematic bottom boundary condition [11]. In fact, one can show that for potential flows the relation between  $\Psi$  and  $\Phi$  is similar to the electromagnetic duality (Appendix).

The dynamics of this system are determined by the Lagrangian density; see [e.g., 9, 10, 11]

$$\mathcal{L} = \phi [\eta_t + \nabla \cdot \Psi + \Psi_z \cdot \nabla \eta]_\eta + \frac{1}{2} \int_{-h}^{\eta} [|\Psi_z|^2 + (\nabla \cdot \Psi)^2] dz - \frac{g}{2} \eta^2, \quad (6)$$

where  $\phi(\mathbf{x}, t)$  is a Lagrange multiplier which ensures that the free-surface kinematic condition is satisfied. The form (6) can be simplified significantly. Using the total derivative  $D_\eta$  (4) and the identity (up to total derivatives)  $\phi D_\eta \cdot \Psi = - (D_\eta \phi) \cdot \Psi = - (\nabla \phi) \cdot \Psi$ , the Lagrangian can be written as:

$$L = \int \mathcal{L} d^2x; \quad \mathcal{L} = \phi \eta_t - \nabla \phi \cdot \Psi + \frac{1}{2} \int_{-h}^{\eta} [|\Psi_z|^2 + (\nabla \cdot \Psi)^2] dz - \frac{g}{2} \eta^2. \quad (7)$$

The vertically integrated term of the Lagrangian (7) may be rewritten as:

$$L_{vert} = \frac{1}{2} \int \int d^2x \int_{-h}^{\eta} [|\Psi_z|^2 + (\nabla \cdot \Psi)^2] dz = \frac{1}{2} \iint d^2x \int_{-h}^{\eta} dz [\partial_i \Theta^i - \Psi \cdot (\partial_z^2 \Psi + \nabla (\nabla \cdot \Psi))] \quad (8)$$

where

$$\Theta = \Psi \nabla \cdot \Psi + (\Psi \cdot \Psi_z) \hat{\mathbf{z}}, \quad (9)$$

is a 3-dimensional vector with divergence. To prove the second equality in (8) note that:

$$\begin{aligned} \partial_i \Theta^i &= \nabla \cdot (\Psi \nabla \cdot \Psi) + (\Psi \cdot \Psi_z)_z \\ &= (\nabla \cdot \Psi)^2 + |\Psi_z|^2 + \Psi \cdot (\partial_z^2 \Psi + \nabla (\nabla \cdot \Psi)). \end{aligned}$$

Because  $\Theta = 0$  on  $z = -h$ , applying Gauss's theorem yields

$$L_{vert} = \frac{1}{2} \iint_{\eta} \Theta \cdot d\mathbf{S} - \frac{1}{2} \iint d^2x \int_{-h}^{\eta} \Psi \cdot (\partial_z^2 \Psi + \nabla \nabla \cdot \Psi) dz,$$

where  $d\mathbf{S} = \mathbf{n} dA$ , with  $dA$  the measure of the area and  $\mathbf{n}$  the normal to the free surface. The first term represents an integral over the free surface and the second an integral over the interior of the fluid. The least action principle

$$\delta \int L dt = 0$$

requires that solutions minimize the action under all possible variations of the fields. In particular, we may consider variations under which the surface terms change independently from the interior terms. This implies that their variations must vanish independently. The variation of the interior term leads to the “Laplace”-like equation:

$$\Psi_{zz} + \nabla(\nabla \cdot \Psi) = 0 \quad (10)$$

This translates to the statement that the two horizontal components of vorticity are zero. We impose this condition as a constraint. The above equation does not fully determine the stream function,  $\Psi$ , which must be fixed by specifying the surface values,  $\Psi(\eta)$  and  $\Psi_z(\eta)$ . The least action principle demands that we minimize under all possible configurations of  $\Psi$ , and, (after imposing the constraint) these configurations are labeled uniquely by  $\Psi(\eta)$  and  $\Psi_z(\eta)$ . Now, imposing Eq. (10), and writing

$$d\mathbf{S} = \frac{(-\nabla\eta, 1)}{\sqrt{1 + |\nabla\eta|^2}} dA; \text{ with } dA = \sqrt{1 + |\nabla\eta|^2} d^2x, \quad (11)$$

one finds the following simplified expression for the interior contribution to the action written entirely in terms of surface functions:

$$L_{vert} = \frac{1}{2} \iint_{\eta} d^2x \Psi \cdot (\Psi_z - \nabla\eta(\nabla \cdot \Psi)), \quad (12)$$

This yields a very simple expression for the Lagrangian density:

$$\mathcal{L} = \frac{1}{2} \Psi^j \mathbf{K}_{jl} \Psi^l - \nabla\phi \cdot \Psi + \phi\eta_t - \frac{g}{2} \eta^2; \text{ with } \mathbf{K}_{jl} = (\delta_{jl}\partial_z - \partial_j\eta\partial_l). \quad (13)$$

where  $\delta_{jl}$  is the Kronecker symbol.

One may think of this as a matrix problem with  $\Psi$  being a vector,  $\mathbf{K}$  being a matrix, and the integral representing a contraction of indices. The quantity  $\Psi^j \mathbf{K}_{jl} \Psi^l$  is equivalent, by definition to  $\Psi^j \{\mathbf{K}_{jl}\}^T \Psi^l$ , where  $\{\mathbf{K}_{jl}\}^T$  is the transpose of  $\mathbf{K}_{jl}$ . Since  $\Psi^j \mathbf{K}_{jl} \Psi^l$  and  $\Psi^j \{\mathbf{K}_{jl}\}^T \Psi^l$  are equal, we may add them and divide by two. For understanding the variations of Lagrangian (13) with respect to  $\Psi$ , it will be convenient to recast it into a symmetric form by formally introducing a new operator  $\mathbf{T} = \frac{1}{2} (\mathbf{K} + \mathbf{K}^T)$ , where  $\mathbf{K}^T$  is the transpose of  $\mathbf{K}$ , i.e.,

$$\mathcal{L} = \frac{1}{2} \Psi^j \mathbf{T}_{jl} \Psi^l - \nabla\phi \cdot \Psi + \phi\eta_t - \frac{g}{2} \eta^2. \quad (14)$$

The parts of  $\mathbf{K}$  that are asymmetric under the transpose operation drop out of the integral after this transform; and  $\mathbf{T}$  allows us to write the constraint in terms of a single linear operator.

When applying the calculus of variation, all functions are varied independently at each point (ignoring for boundary conditions restricting the variation for now). Thus, we may consider a variation of the stream function in one unit of volume completely independently of neighboring units of volume. Then we may conclude that all solutions must satisfy the interior equation (10) without having to solve all the equations involving the surface terms. After requiring equation (10) to hold, we still need to specify  $\Psi$  and  $\Psi_z$  in order to determine a solution uniquely. Thus, we may think of  $\Psi$  and  $\Psi_z$  as labeling the full solution so that we have just traded variation over the complete set of functions with variations on a reduced set of functions which already satisfy the interior equations. The value of  $\Psi$  is then determined by these remaining variations. The governing equations are given by the variations of the Lagrangian (14) with respect to the variables  $\eta$ ,  $\phi$ , and  $\Psi$ , respectively:

$$\eta_t + D_\eta \cdot \Psi = 0 \quad \text{on} \quad z = \eta \quad (15)$$

$$\phi_t + \nabla \cdot (\phi \Psi_z) - \frac{1}{2} \left[ |\Psi_z|^2 + (\nabla \cdot \Psi)^2 \right] + g\eta = 0 \quad \text{on} \quad z = \eta \quad (16)$$

$$\begin{aligned} & [\Psi_z - (\nabla \cdot \Psi) \nabla \eta + \nabla(\Psi \cdot \nabla \eta) - 2 \nabla \phi + (\nabla \eta \cdot \Psi_z) \nabla \eta] \cdot \delta \Psi \\ & + [\Psi + (\nabla \eta \cdot \Psi) \nabla \eta] \cdot \delta \Psi_z = 0 \quad \text{on} \quad z = \eta \end{aligned} \quad (17)$$

Note that the variation  $\delta \Psi_z$  is treated as independent of  $\delta \Psi$  since we have a surface boundary and total derivatives may not be discarded arbitrarily. (Or, one may note that both  $\Psi$  and  $\Psi_z$  must be independently specified in order to determine a solution to the second order equation (10).) Equation (17) really is a vector equation since the two components of  $\delta \Psi$  may be varied independently. After applying the variation to  $\delta \Psi$  and  $\delta \Psi_z$ , we will obtain two vector equations. Equations (15-17) are exact conditions evaluated at the surface that determine  $\eta$ ,  $\phi$ , and  $\Psi$  to arbitrary degree of accuracy. No approximations have been made so far concerning the interior solutions. Together with the ‘‘Laplace-like’’ equation, (10), equations (15-17) form the governing equations for the  $\Psi$ -formalism, with unknown functions:  $\eta(\mathbf{x}, t)$ ,  $\phi(\mathbf{x}, t)$ , and  $\Psi(\mathbf{x}, z, t)$ .

As will be shown later,  $\eta$  and  $\phi$  remain canonically-conjugated variables, which is useful for deriv-

ing a Hamiltonian description of the flow. Equations (15-16) are therefore dynamical equations for  $\eta$  and  $\phi$ , while (10) constrains the vertical structure of the flow. (17) provides a relation between the two dynamic variables via the function  $\Psi$ .

The first step toward solving the dynamical equations (15-16) is to eliminate  $\Psi$  using the constraint equation (17). Note that taking the variation of (14) with respect to  $\Psi$  gives

$$\mathbf{T}\Psi - \nabla\phi = 0 \quad \text{on} \quad z = \eta. \quad (18)$$

with the formal solution

$$\Psi = \mathbf{T}^{-1}\nabla\phi \quad \text{on} \quad z = \eta. \quad (19)$$

Equation (19) is a formal representation of complicated dynamics:  $\mathbf{T}$  is defined on the set of 2-dimensional vector fields  $\Psi(\mathbf{x}, z = \eta, t)$  satisfying the interior equation (10) and the bottom boundary condition (5). Inverting  $\mathbf{T}$  therefore means inverting within the image of this space of functions. It may be shown that  $\mathbf{T}$  is invertible on this space and provides a solution to the constraint. Without further elaboration on this point (will be presented elsewhere), we note that important information may be gleaned from equation (19): 1) We see that each term in a perturbative expansion for  $\Psi$  will contain only one power of  $\phi$  and an arbitrary number of  $\eta$ ; and 2) the variation with respect to  $\Psi$  will produce a total derivative in an effective surface Lagrangian for  $\phi$  and  $\eta$ . In the present work, we will not pursue the explicit construction of  $\mathbf{T}$ , rather, we will work directly with the equivalent equation (17).

We conclude this section by noting that retaining only the quadratic terms of the constraint equation (17), the linear theory of [10] is retrieved, where

$$\Psi_z = \nabla\phi + \text{higher order terms.}$$

In the leading order,  $\phi$  is equal to the velocity potential at the surface, although it seems difficult to provide a more intuitive statement. The physically meaningful variable is the surface elevation  $\eta$ ; and  $\phi$  is simply the variable that is canonically conjugate to that. In the full theory, this interpretation is corrected by higher order terms.

### 3. Homogeneous flows over slowly varying topography

The formalism presented above provides a consistent means for incorporating both varying topography and vertical vorticity. We illustrate here the application of the theory to flows that admits

a wave-number Fourier representation. For simplicity, the discussion will be limited to slowly varying topography. The analysis of more complicated settings will be presented elsewhere.

### 3.1. Interior solutions

Assuming that the problem is horizontally homogeneous, the unknown functions admit wave number Fourier representation

$$\begin{pmatrix} \eta \\ \phi \\ \Psi \end{pmatrix} = \int \frac{d^2k}{2\pi} \begin{pmatrix} \eta_{\mathbf{k}} \\ \phi_{\mathbf{k}} \\ \Psi_{\mathbf{k}} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \begin{pmatrix} \eta_{\mathbf{k}} \\ \phi_{\mathbf{k}} \\ \Psi_{\mathbf{k}} \end{pmatrix} = \int d^2x \begin{pmatrix} \eta \\ \phi \\ \Psi \end{pmatrix} e^{-i\mathbf{k}\cdot\mathbf{x}} \quad (20)$$

where  $\mathbf{k} = k^j \hat{\mathbf{x}}_j$  is the wave number vector, and the  $g_{\mathbf{k}} = [g]_{\mathbf{k}}$  is the Fourier transform of  $g$ . The short-hand notation  $[\dots]_{\mathbf{k}}$  will later simplify the handling of convolution products resulting from the Fourier transform of nonlinear terms. Because the functions  $\eta$  and  $\phi$  are real, their transforms satisfy the regular symmetry conditions, e.g.,  $\eta_{\mathbf{k}} = \eta_{-\mathbf{k}}^*$  with the asterisk denoting the complex conjugate. In Fourier space, the Helmholtz-Hodge decomposition [e.g., 15] of the flow in terms of  $\Psi$  is

$$\Psi = \theta + \Gamma; \quad \begin{pmatrix} \theta \\ \Gamma \end{pmatrix} = \int \frac{d^2k}{2\pi} \begin{pmatrix} \theta_{\mathbf{k}} \hat{\mathbf{k}} \\ \Gamma_{\mathbf{k}} \hat{\mathbf{z}} \times \hat{\mathbf{k}} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (21)$$

where  $\theta$  and  $\Gamma$  are longitudinal and transversal components representing the curl-free and the divergence-free motions. It may be checked that  $\nabla \times \theta = \nabla \cdot \Gamma = 0$ , and also  $\text{curl}_3 \mathbf{u}_\theta = \text{div}_3 \mathbf{u}_\Gamma = 0$ , where  $\mathbf{u}_{\theta,\gamma}$  are the velocity components defined through equation (2), and  $\text{curl}_3$  and  $\text{div}_3$  are the 3-dimensional versions of the operators. Both components satisfy the required Fourier symmetries for real physical domain functions. Therefore we obtain  $\nabla(\nabla \cdot \Gamma) = 0$  and  $\nabla(\nabla \cdot \theta) = \int \frac{d^2k}{2\pi} (-k^2 \theta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}})$ . Substituting decomposition (21) into the governing equation (10) for the interior flow yields

$$\partial_z^2 \theta_{\mathbf{k}} - k^2 \theta_{\mathbf{k}} = 0; \quad \partial_z^2 \Gamma_{\mathbf{k}} = 0, \quad (22)$$

with  $k$  ( $k^2 = k^j k^j$ ) the absolute value of the wave number. In the physical domain the first equation is simply the Laplace equation for the curl-free component of the flow. For mildly sloping bottoms (e.g., [16]), the solution of the equation for  $\theta_{\mathbf{k}}$  is the usual

$$\theta_{\mathbf{k}} = \frac{\sinh[k(z+h)]}{\sinh(kh)} \vartheta_{\mathbf{k}}, \quad (23)$$



where the wave number is assumed to be a slowly varying function of the horizontal coordinate. The second equation produces trivial linear solutions, suitable for describing sheared currents. Explicitly, we shall take  $\Gamma_{\mathbf{k}}$  to be of the form:

$$\Gamma_{\mathbf{k}} = \frac{z + h'}{h'} \gamma_{\mathbf{k}}. \quad (24)$$

Note that the actual velocity is obtained by taking a  $z$  derivative of the streamfunction. Thus, a linear term in  $\Gamma$  translates to a velocity that is constant in depth as required.

Impermeability is automatically satisfied in this formalism so long as  $\Psi(z = -h) = 0$ , as may be seen by equation (5). Since equations (20)-(24) are consistent with  $\Psi(z = -h) = 0$ , there is no violation of the impermeability condition. Here, we use an  $h'$  rather than an  $h$  to leave open the possibility that the flow represented by  $\Gamma$  terminates at some finite depth which is not necessarily the actual bottom. (This phenomenon is observed in surface currents in the open ocean.) For simplicity, we shall henceforth assume that  $h' = h$ .

The vertical vorticity is an unambiguous physical quantity and it must be specified as an input or initial condition for the particular application in mind. For any choice of initial velocity profile one may solve for the relevant  $\Gamma_{\mathbf{k}}$  by inverting equations (21) and (24). The dynamical equations then specify the future evolution of the system. For the readers convenience we invert (21) in order to provide the expression for  $\gamma_{\mathbf{k}}$  in terms of the observed or modeled velocity profile.

$$h' \int d^2\mathbf{x} (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \cdot \mathbf{u} e^{-i\mathbf{k} \cdot \mathbf{x}} = \gamma_{\mathbf{k}} \quad (25)$$

### 3.2. Perturbation solution

A common approach to seek solutions for surface-gravity wave equations such as, (10), and (15-18), is to expand them in powers of  $\eta$ . The governing equations may either be expanded directly, or equivalently, re-derived from the expanded Lagrangian. Keeping up to cubic terms in the unknown functions yields the following expansions

$$\eta_t + \nabla \cdot \left( \Psi + \eta \Psi_z + \frac{1}{2} \eta^2 \Psi_{zz} \right) = 0 \text{ on } z = 0, \quad (26)$$

$$\begin{aligned} \phi_t + \nabla \phi \cdot \Psi_z + \eta \nabla \phi \cdot \Psi_{zz} - \frac{1}{2} \left[ |\Psi_z|^2 + (\nabla \cdot \Psi)^2 \right] \\ - \frac{1}{2} \eta \left[ (\Psi_z)^2 + (\nabla \cdot \Psi)^2 \right]_z + g\eta = 0 \text{ on } z = 0, \end{aligned} \quad (27)$$

$$\begin{aligned} \left[ \Psi_z - \nabla \phi - \nabla(\eta \nabla \cdot \Psi) - \frac{1}{2} \nabla(\eta^2 \nabla \cdot \Psi_z) \right] \delta \Psi + \eta \{ \Psi_z - \nabla \phi \\ - \nabla \eta (\nabla \cdot \Psi) - \eta [\nabla(\nabla \cdot \Psi)] \} \delta \Psi_z + \frac{1}{2} \eta^2 (\Psi_z - \nabla \phi) \delta \Psi_{zz} = 0, \text{ on } z = 0 \end{aligned} \quad (28)$$

where terms containing fourth- (or higher) order products of  $\eta$ ,  $\phi$ , and  $\Psi$  have been neglected.

A solution for  $\Psi$  in terms of  $\eta$  and  $\phi$  can be obtained by substituting the form given by equation (23) into the constraint equation (28) and considering variations of  $\vartheta$  and  $\gamma$  separately. After some algebra, separating the curl-free and divergence-free components, and neglecting terms of order higher than cubic, the constraint equation (28) yields

$$\vartheta_{\mathbf{k}} + \int \frac{dk dk_1}{2\pi} \left[ (m_{11})_{\mathbf{k}-\mathbf{k}_1} \vartheta_{\mathbf{k}_1} + (m_{12})_{\mathbf{k}-\mathbf{k}_1} \gamma_{\mathbf{k}_1} \right] = F_1, \quad (29)$$

$$\gamma_{\mathbf{k}} + \int \frac{dk dk_1}{2\pi} \left[ (m_{21})_{\mathbf{k}-\mathbf{k}_1} \vartheta_{\mathbf{k}_1} + (m_{22})_{\mathbf{k}-\mathbf{k}_1} \gamma_{\mathbf{k}_1} \right] = F_2, \quad (30)$$

where  $\mathbf{k}$ ,  $\mathbf{k}_1$  are wave numbers. We use the following short-hand conventions:  $[\dots]_{\mathbf{k}}$  for the Fourier transform (20);  $\text{th}kh$  for  $\tanh kh$ ; and  $\text{cth}kh$  for  $\coth kh$ . The coefficients of the left-hand side are

$$\begin{aligned} (m_{11})_{\mathbf{k},\mathbf{k}_1} &= [\mathbf{k}_1 \cdot \hat{\mathbf{k}} \text{cth}(k_1 h) + k_1 \text{th}(k_1 h)] \eta_{\mathbf{k}_1} \\ &\quad + \frac{1}{2} [k_1 \mathbf{k}_1 \cdot \hat{\mathbf{k}} + k k_1 + \text{th}(kh) \text{cth}(k_1 h) (\mathbf{k}_1 \cdot \mathbf{k} + k_1^2)] [\eta^2]_{\mathbf{k}-\mathbf{k}_1} \\ (m_{12})_{\mathbf{k},\mathbf{k}_1} &= \frac{1}{h} (\hat{\mathbf{z}} \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{k}} \eta_{\mathbf{k}-\mathbf{k}_1} + \frac{1}{2h} k \text{th}(kh) (\hat{\mathbf{z}} \times \hat{\mathbf{k}}_1) \cdot \hat{\mathbf{k}} [\eta^2]_{\mathbf{k}-\mathbf{k}_1} \\ (m_{21})_{\mathbf{k},\mathbf{k}_1} &= \mathbf{k}_1 \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \text{cth}(k_1 h) \eta_{\mathbf{k}-\mathbf{k}_1} + \frac{1}{2} k_1 \mathbf{k}_1 \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) [\eta^2]_{\mathbf{k}-\mathbf{k}_1} \\ (m_{22})_{\mathbf{k},\mathbf{k}_1} &= \frac{1}{h} \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}} \eta_{\mathbf{k}-\mathbf{k}_1} \end{aligned} \quad (31)$$

and the right-hand side terms are

$$\begin{aligned}
F_1 &= i (\text{th}kh) \phi_{\mathbf{k}} + i \int \frac{dkdk_1}{2\pi} \mathbf{k}_1 \cdot \hat{\mathbf{k}} \eta_{\mathbf{k}-\mathbf{k}_1} \phi_{\mathbf{k}_1} + \frac{i}{2} \int \frac{dkdk_1}{2\pi} \mathbf{k}_1 \cdot \mathbf{k} \text{th}kh [\eta^2]_{\mathbf{k}-\mathbf{k}_1} \phi_{\mathbf{k}_1}, \\
F_2 &= i \iint \frac{dkdk_1}{2\pi} k_1 \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \eta_{\mathbf{k}-\mathbf{k}_1} \phi_{\mathbf{k}_1}.
\end{aligned} \tag{32}$$

Equations (29-30) are linear in  $\vartheta$  and  $\gamma$  to any order in the nonlinearity and may be solved in terms of  $\phi$  and  $\eta$ . In a matrix form, the system (29-30) is

$$(I + M) \psi = F, \text{ with, } \psi = \begin{pmatrix} \vartheta \\ \gamma \end{pmatrix}, M_{jl} \psi^j = \int \frac{dkdk_1}{2\pi} (m_{jl})_{\mathbf{k}, \mathbf{k}_1} (\psi^j)_{\mathbf{k}_1}, \tag{33}$$

where  $I$  is the identity matrix and  $j, l = 1, 2$ . Because the elements of the matrix  $M$  are higher ordered, equation (33) may be inverted directly to the order of accuracy required as

$$\psi = (I - M + M^2 - \dots) F \tag{34}$$

The procedure to solve for  $\vartheta$  and  $\gamma$  is now straightforward, albeit tedious. After some algebra, one finds the following solutions up to third order terms:

$$\begin{aligned}
\vartheta_{\mathbf{k}} &= i \text{th}(kh) \phi_{\mathbf{k}} - i \int \frac{dkdk_1}{2\pi} k_1 \text{th}(kh) \text{th}k_1 h \eta_{\mathbf{k}-\mathbf{k}_1} \phi_{\mathbf{k}_1} \\
&\quad + i \int \frac{dkdk_1 dk_2}{(2\pi)^2} W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \eta_{\mathbf{k}-\mathbf{k}_1} \eta_{\mathbf{k}_1-\mathbf{k}_2} \phi_{\mathbf{k}_2},
\end{aligned} \tag{35}$$

$$\gamma_{\mathbf{k}} = i \frac{dkdk_1 dk_2}{(2\pi)^2} k_1 \text{th}(k_1 h) \left( \mathbf{k} - \frac{1}{2} \mathbf{k}_1 \right) \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{k}}) \eta_{\mathbf{k}-\mathbf{k}_1} \eta_{\mathbf{k}_1-\mathbf{k}_2} \phi_{\mathbf{k}_2} \tag{36}$$

with the coefficient

$$\begin{aligned}
W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} &= k_2 [\mathbf{k}_1 \cdot \hat{\mathbf{k}} + k \text{th}(kh) \text{th}(k_1 h)] \text{th}(k_2 h) \\
&\quad - \frac{1}{2} (k_2 \mathbf{k}_2 \cdot \hat{\mathbf{k}} + k k_2) \text{th}(k_2 h) - \frac{1}{2} k^2 \text{th}(kh).
\end{aligned} \tag{37}$$

Remarkably, the inclusion of shear does not affect the solution for  $\vartheta$  to this order. Furthermore, it will become apparent that  $\gamma$  itself does not contribute to 4-wave interactions, since its contribution is eliminated by vector identities at this order. Thus, shear does not affect 3- and 4-wave interactions in an isotropic background.  $\gamma$  becomes significant with the inclusion of a strong shear as a background forcing in the Lagrangian, which will be pursued in a future work.

## 4. Hamiltonian and Lagrangian formalisms

Within the assumptions made in the development of the approach presented here, it is possible to derive Hamiltonian and Lagrangian formalism. These allow for direct comparison with existing Hamiltonian theories and will provide a simpler basis for applications, based on observable quantities.

### 4.1. Hamiltonian formulation

The derivation of a Hamiltonian formalism starts by noting that, as in the potential formulation for the linear problem,  $\eta$  and  $\phi$  are canonical variables. For the  $\Psi$ -formalism, equation (6) shows that  $\frac{dL}{d(\eta_t)} = \phi$ . The only question is whether or not there is some additional ‘hidden’ dependence on  $\eta_t$  via  $\Psi$ . However, the explicit equations for  $\Psi$  (e.g., constraint (17)) show that  $\Psi$  is independent on  $\eta_t$ . Thus, the usual argument that  $\phi$  and  $\eta$  are canonically-conjugated variables follows in this formalism as well.

Indeed, the conjugate momentum of the dynamical variable  $\eta$  is

$$\phi = \frac{\partial \mathcal{L}}{\partial \eta_t}, \quad (38)$$

as can be seen from the definition (14) of the Lagrangian, and noting that  $\Psi$  has no explicit dependence on  $\eta_t$  (e.g., solution (19)). The Legendre transformation

$$H = \int d^2x \mathcal{H} = \int d^2x (\phi \eta_t - \mathcal{L}) \quad (39)$$

then yields

$$\mathcal{H} = \nabla \phi \cdot \Psi - \frac{1}{2} \Psi \cdot (\Psi_z - \nabla \eta (\nabla \cdot \Psi)) + \frac{g}{2} \eta^2 \quad (40)$$

where  $\Psi$  is given by equation (19), written in terms of  $\phi$  and  $\eta$ . An explicit form for the Hamiltonian is obtained by substituting equations (35) and (36) into (40). As mentioned before, (36) does not contribute at this order. Using the symmetric form of convolution products, the results in [18]

are retrieved exactly to order  $O(\varepsilon^5)$

$$\begin{aligned}
H &= H_2 + H_3 + H_4 + O(\varepsilon^5), \\
H_2 &= \frac{1}{2} \int dk \left( k \operatorname{th}(kh) |\phi_{\mathbf{k}}|^2 + g |\eta_{\mathbf{k}}|^2 \right), \\
H_3 &= \frac{1}{2} \int \frac{dk_1 dk_2 dk_3}{2\pi} T_{\mathbf{k}_1 \mathbf{k}_2}^{(1)} \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \\
H_4 &= \frac{1}{2} \int \frac{dk_1 dk_2 dk_3 dk_4}{4\pi^2} T_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(2)} \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \eta_{\mathbf{k}_3} \eta_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4),
\end{aligned} \tag{41}$$

where  $\delta$  is the Dirac delta, and the interaction coefficients in symmetric form are

$$\begin{aligned}
T_{\mathbf{k}_1 \mathbf{k}_2}^{(1)} &= -\mathbf{k}_1 \cdot \mathbf{k}_2 - |\mathbf{k}_1| |\mathbf{k}_2| \operatorname{th}(k_1 h) \operatorname{th}(k_2 h), \\
T_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(2)} &= -\frac{1}{2} k_1 k_2^2 \operatorname{th}(k_1 h) - \frac{1}{2} k_2 k_1^2 \operatorname{th}(k_2 h) \\
&\quad + \frac{1}{4} k_1 k_2 |\mathbf{k}_1 + \mathbf{k}_3| \operatorname{th}(k_1 h) \operatorname{th}(k_2 h) \operatorname{th}(|\mathbf{k}_1 + \mathbf{k}_3| h) \\
&\quad + \frac{1}{4} k_1 k_2 |\mathbf{k}_2 + \mathbf{k}_3| \operatorname{th}(k_1 h) \operatorname{th}(k_2 h) \operatorname{th}(|\mathbf{k}_2 + \mathbf{k}_3| h) \\
&\quad + \frac{1}{4} k_1 k_2 |\mathbf{k}_1 + \mathbf{k}_4| \operatorname{th}(k_1 h) \operatorname{th}(k_2 h) \operatorname{th}(|\mathbf{k}_1 + \mathbf{k}_4| h) \\
&\quad + \frac{1}{4} k_1 k_2 |\mathbf{k}_2 + \mathbf{k}_4| \operatorname{th}(k_1 h) \operatorname{th}(k_2 h) \operatorname{th}(|\mathbf{k}_2 + \mathbf{k}_4| h).
\end{aligned} \tag{42}$$

#### 4.2. Lagrangian formulation

Based on the expansion described above, we are seeking here a Lagrangian description based on observable quantities, i.e., the generalized coordinate  $\eta$  and generalized velocity  $\partial_t \eta$ . The first step is to eliminate  $\phi$  by solving its equation of motion (15) for  $\Psi$  in terms of  $\eta$  and  $\partial_t \eta$ . The fact that  $\gamma$  will not contribute at quartic order gives us the freedom to ignore  $\gamma$  and just solve for  $\vartheta$ . Note that this does not preclude, from including background vorticity in the solution of  $\vartheta_{\mathbf{k}}$ . Following a similar procedure as before yields

$$\begin{aligned}
\vartheta_{\mathbf{k}} &= \frac{i}{k} (\partial_t \eta_{\mathbf{k}}) - i \int \frac{dk dk_1}{2\pi} [\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_1 \operatorname{cth}(k_1 h)] \eta_{\mathbf{k}-\mathbf{k}_1} (\partial_t \eta_{\mathbf{k}_1}) \\
&\quad + i \int \frac{dk dk_1 dk_2}{(2\pi)^2} V_{\mathbf{k} \mathbf{k}_1 \mathbf{k}_2} \eta_{\mathbf{k}-\mathbf{k}_1} \eta_{\mathbf{k}_1-\mathbf{k}_2} (\partial_t \eta_{\mathbf{k}_2})
\end{aligned} \tag{43}$$

with the coefficient

$$V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_1) (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2) \text{cth}(k_1 h) \text{cth}(k_2 h) - \frac{1}{2} \hat{\mathbf{k}} \cdot \mathbf{k}_2. \quad (44)$$

Substituting this back into equation (13) (via (23), and (20)) and ignoring the  $\phi$  constraint, the Lagrangian, valid up to quartic order, reads

$$\begin{aligned} L &= L_2 + L_3 + L_4 + O(\varepsilon^5), \\ L_2 &= \frac{1}{2} \int d^2 k \left[ \frac{\text{cth}(kh)}{k} |\partial_t \eta_{\mathbf{k}}|^2 - \frac{g}{2} |\eta_{\mathbf{k}}|^2 \right] \\ L_3 &= \frac{1}{2} \int \frac{dk_1 dk_2 dk_3}{2\pi} G_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(1)} (\partial_t \eta_{\mathbf{k}_1}) (\partial_t \eta_{\mathbf{k}_2}) \eta_{\mathbf{k}_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ L_4 &= \frac{1}{2} \int \frac{dk_1 dk_2 dk_3 dk_4}{4\pi^2} G_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(2)} (\partial_t \eta_{\mathbf{k}_3}) (\partial_t \eta_{\mathbf{k}_4}) \eta_{\mathbf{k}_1} \eta_{\mathbf{k}_2} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \end{aligned} \quad (45)$$

with the interaction coefficients

$$G_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(1)} = (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \text{cth}(k_1 h) \text{cth}(k_2 h) \quad (46)$$

$$\begin{aligned} G_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(2)} &= \text{cth}(k_1 h) \left\{ \left[ |\mathbf{k}_1| - \hat{\mathbf{k}}_1 \cdot \mathbf{k}_2 + 2|\mathbf{k}_1 + \mathbf{k}_4| \left( \frac{\mathbf{k}_1 + \mathbf{k}_4}{|\mathbf{k}_1 + \mathbf{k}_4|} \right) \cdot \hat{\mathbf{k}}_1 \right] \right. \\ &\quad \left. + \left[ \left( \frac{\mathbf{k}_1 + \mathbf{k}_4}{|\mathbf{k}_1 + \mathbf{k}_4|} \right) \cdot \hat{\mathbf{k}}_1 \right] \left[ \left( \frac{\mathbf{k}_1 + \mathbf{k}_4}{|\mathbf{k}_1 + \mathbf{k}_4|} \right) \cdot \hat{\mathbf{k}}_2 \right] |\mathbf{k}_1 + \mathbf{k}_4| \text{cth}(k_2 h) \text{cth}(|\mathbf{k}_1 + \mathbf{k}_4| h) \right\}. \end{aligned} \quad (47)$$

One could make it symmetric by switching indices. It is easy to check that  $L_2$  is consistent with the results given by Zakharov [13] via performing a Legendre transform of (41).  $L_3$  also agrees with Zakharov's formalism if we substitute the first order relationship between  $\partial_t \eta$  and  $\phi$  into (42). Following the same procedure,  $L_4$  shows consistency to (42) in 1-D situations (e.g.,  $\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_4$  and  $\hat{\mathbf{k}}_2, \hat{\mathbf{k}}_3$  are in the opposite direction). The dot products in  $G_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}^{(2)}$  is due to the nature of streamfunction vector. This may lead to different consequences for interactions of directional waves.

## 5. Summary

The original formulation of the  $\Psi$ -formalism [9, 10] was derived under the constraints of irrotational flows and a homogeneous bottom topography. This study investigates the consequences of removing these constraints and presents a theory based on the  $\Psi$ -formalism.

One convenient feature of the classic potential ( $\Phi$ ) formalism is that the interior flow structure may be completely eliminated using its equation of motion and boundary conditions on the surface.

The dynamics may be described by a Hamiltonian function written entirely in terms of the surface variables, i.e., free-surface displacement  $\eta$  and its canonical conjugate, the surface velocity potential  $\phi$ . In reformulating the  $\Psi$ -formalism, the approach for developing the  $\Phi$ -formalism is applied to derive a new form of the Lagrangian first proposed by Kim et al. [2001, 2003], based only on surface variables. This allows for separating problems related to interior approximations from the issue of expressing  $\Psi$  in terms of  $\phi$  and  $\eta$ . The governing equations that result from taking variations of the Lagrangian with respect to the unknown functions are organized in terms of dynamical equations for the canonically-conjugate pair  $(\eta, \phi)$ , a surface constraint equation that provides a connection between  $\Psi$  and  $(\eta, \phi)$ , and the interior equation (essentially a condition that cancels two horizontal components of the vorticity). For horizontally homogeneous flows over mild topography, the Helmholtz-Hodge decomposition allows for an asymptotic solution for the constraint equation, thus providing the basis for deriving a surface Hamiltonian and Lagrangian. Expanding these in terms of  $\eta$  around a stationary background, we conclude that the  $\Psi$ -formalism leads to a Hamiltonian identical to the standard potential formalism [18] for terms up to fourth order. Furthermore, this formalism yields a new quartic Lagrangian written in directly observable quantities  $\eta$  and  $\partial_t \eta$ . The description allows wave to propagate across background flows with vertical vorticity. Applications of this feature will be presented in future work.

### Appendix. The $\Psi - \Phi$ duality for potential flows

A curl-free (zero-vorticity) flow  $(\mathbf{u}, w)$  is fully determined by the potential function  $\Phi(\mathbf{x}, z, t)$  as

$$\mathbf{u} = \nabla \Phi, \quad \text{and} \quad w = \Phi_z, \quad (48)$$

or, using differential forms [e.g., 19], as

$$d\Phi = u^j dx_j + w dz, \quad (49)$$

with  $\Phi$  a 0-form. For such a flow, the streamfunction  $\Psi$  (1) provides an alternative description. Because both  $\Psi$  and  $\Phi$  completely define the flow, they determine each other up to a total derivative. For potential flows, this relationship is analogous to the so-called electromagnetic (EM) duality [20]. Note that this duality exists only for potential flows, since the  $\Phi$  does not completely determine flows with non-zero vorticity.

Based on  $\Phi$ , the EM duality defines an 3-vector  $\Psi'$  as

$$*d\Phi = d\Psi', \quad (50)$$

where the star denotes the Hodge dual. The above relation defines the 1-form  $\Psi'$  through its differential, hence up to a differential of a 0-form. Since the differential operator  $d$  satisfies  $d^2 = 0$ , one can use the 0-form  $\sigma$  to shift  $\Psi'$  by:  $\Psi' \rightarrow \Psi' + d\sigma$  (gauge invariance). Without loss of generality, this allows to set the  $z$ -component  $(\Psi')^z = 0$ , which we shall henceforth assume. The 1-form  $\Psi'$  is now a 2-vector. One may easily check that a  $90^\circ$  rotation in the  $\mathbf{x}$  plane produces a 2-vector  $\Psi$  that is precisely the streamfunction defined by (2).

As is the case with EM duality, a rigid constraint in one of the variables may become a soft constraint with the other. By applying the differential operator directly to Eq. (50), one sees that

$$d*d\Phi = d^2\Psi' \quad \Rightarrow \quad \Delta\Phi = d^2\Psi'.$$

The right hand side is zero as a mathematical identity whereas the left hand side is zero as the result of a condition on  $\Phi$ . The physical meaning of both sides is that the fluid is incompressible. Similarly, by applying first the Hodge dual to both sides of (50) and then the differential operator, we obtain

$$d^2\Phi = d*d\Psi'.$$

This is the condition of zero vorticity and it gives the condition under which a mapping from  $\Psi$  into  $\Phi$  exists. As before, a constraint for one variable,  $\Phi$ , becomes a Laplace equation for the dual variable,  $\Psi'$ .

In the  $\Psi$ -formalism, the right hand side of the above equation is not required to hold. Recall that we first use the gauge invariance to set  $(\Psi')^z = 0$ . If we take the variation of the kinetic term  $\frac{1}{2}(\mathbf{u}^2 + w^2)$  in the usual definition of the action, without including the term coming from varying  $(\Psi')^z$ , instead of  $d*d\Psi' = 0$ , we simply get

$$\nabla(\nabla \cdot \Psi) + \partial_z^2 \Psi = 0.$$

This is, therefore, a restricted form of ' $d*d\Psi' = 0$ ' which allows for one component of vorticity. As a final note this form of EM duality is non-local, meaning that  $\Phi$  evaluated at a point  $(\mathbf{x}, z)$  may



not be determined from  $\Psi$  and its derivatives at that point alone, instead they are related through an integro-differential transformation. Furthermore, this transformation generally exchanges Neumann and Dirichlet boundary conditions, as we have already seen.

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